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INTEGRAL TRANSFORM SOLUTION FOR THE LID-DRIVEN CAVITY FLOW PROBLEM IN STREAMFUNCTION-ONLY FORMULATION

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SUMMARY

The basic ideas in the generalized integral transform technique are further advanced to allow for the hybrid numerical–analytical solution of the two-dimensional steady Navier–Stokes equations in streamfunction-only formulation. The classical lid-driven square cavity problem is selected for illustration of the approach. The corresponding biharmonic-type non-linear partial differential equation for the streamfunction is integral transformed in one of the co-ordinates and an infinite system of coupled non-linear ODEs for the transformed potential results in the other independent variable. Upon truncation to an appropriate finite order, the ODE system is numerically solved by well-established algorithms with automatic error control devices. The convergence behaviour of the eigenfunction expansion is demonstrated and reference results are provided for typical values of Reynolds number.

KEY WORDS Navier–Stokes equations Cavity flow Analytical solutions Integral transforms

INTRODUCTION

The Navier–Stokes equations model some of the most important problems in the heat and fluid flow field and the last two decades have been dedicated to the development of reliable and accurate solution procedures for this class of problems. The non-linear nature of these models together with the competition between convection and diffusion phenomena, represented by inertia and viscous effects respectively, make the solution of this set of equations still a difficult task even for the best known numerical techniques. Therefore several research groups continue to invest time and effort in the improvement of existing schemes as well as in the development of new approaches, in parallel with computing hardware and numerical analysis progress. These contributions are in general based on classical test cases that allow critical comparisons among the different techniques to a certain extent. A frequently employed problem that models two-dimensional incompressible steady flow situations is the lid-driven square cavity problem, as reviewed in various references.[1–19] Most of the previous work is related to variations and enhancements of the well known finite difference[1–12] and finite element[11–18] methods, in addition to recent implementations of boundary element[19] and finite analytic[11] approaches. Agreement among all such sets of results is far from perfect and truly benchmark results are available only for asymptotic situations ($Re=0$ and $Re\rightarrow\infty$), making it particularly difficult to assess the relative merits of each individual scheme.

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In recent years the so-called generalized integral transform technique[20] has gradually advanced towards the hybrid numerical–analytical solution of *a priori* non-transformable linear diffusion/convection 'problems, based on the formal analytic ideas for the exact solution of different classes of transformable problems.[21] More recently, this approach was quite successfully utilized in the automatic and accuracy-controlled solution of non-linear diffusion and convection–diffusion problems,[20,22–26] including situations of moving boundaries, irregular geometries, non-linear equation and boundary source terms, conjugated and coupled problems, non-linear transport coefficients, non-linear convective terms and boundary layer equations. The next natural step in the establishment of this hybrid approach is the solution of the full Navier–Stokes equations. Therefore the present paper is aimed at advancing the integral transform method to handle this class of problems, here represented by the classical square cavity test case. The streamfunction-only formulation is preferred, since boundary conditions are explicitly provided and the auxiliary eigenvalue-type problem is more easily defined. The related non-linear biharmonic partial differential equation is integral transformed by eliminating one of the space variable's dependence and obtaining an infinite system of coupled non-linear ordinary differential equations for the transformed streamfunctions. For computational purposes the infinite system is truncated to a finite order sufficiently large to achieve the prescribed convergence tolerance. Boundary value problem solvers are then readily available in scientific subroutine libraries,[27] with automatic error control procedures, that provide accuracy-controlled numerical results for the transformed potentials in the direction not eliminated through the integral transformation process. The desired original potential is recovered at any time, in explicit analytic form, by recalling the previously established inversion formula. The convergence behaviour of the proposed eigenfunction expansion is here examined for a few representative values of the governing parameter, the Reynolds number, at different positions within the medium.

ANALYSIS

We consider the two-dimensional steady incompressible laminar flow of a Newtonian fluid inside a square cavity, due to a continuously moving top end wall at a constant velocity, according to Figure 1. The related Navier–Stokes equations in vorticity transport formulation and dimensionless form are written as

$$\frac{1}{Re}\left(\frac{\partial^2\omega(x,y)}{\partial x^2}+\frac{\partial^2\omega(x,y)}{\partial y^2}\right)=\frac{\partial\psi}{\partial y}\frac{\partial\omega}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\omega}{\partial y}, \tag{1a}$$

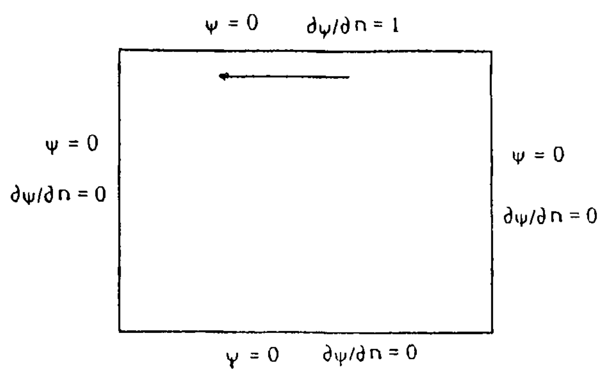

Figure 1. Geometry and co-ordinate system for square cavity problem

$$\omega(x, y) = -\left(\frac{\partial^2 \psi(x, y)}{\partial x^2} + \frac{\partial^2 \psi(x, y)}{\partial y^2}\right) \quad \text{in } 0<x<1, 0<y<1, \tag{1b}$$

where $\omega(x, y)$ is the vorticity, $\psi(x, y)$ is the streamfunction and Re is the Reynolds number. The appropriate boundary conditions are given by

$$\psi(0, y)=0, \qquad \partial\psi(0, y)/\partial x=0, \qquad 0\le y<1, \tag{1c, d}$$

$$\psi(1, y)=0, \qquad \partial\psi(1, y)/\partial x=0, \qquad 0\le y<1, \tag{1e, f}$$

$$\psi(x, 0)=0, \qquad \partial\psi(x, 0)/\partial y=0, \qquad 0\le x\le 1, \tag{1g, h}$$

$$\psi(x, 1)=0, \qquad \partial\psi(x, 1)/\partial y=-1, \quad 0<x<1. \tag{1i, j}$$

All the required boundary conditions are specified in terms of the streamfunction. Therefore it becomes more convenient, especially when choosing the auxiliary eigenvalue-type problem, to rewrite (1a, b) in the so-called streamfunction-only formulation. Then, substituting (1b) into (1a), we find

$$\frac{\partial^4\psi}{\partial x^4} + 2\frac{\partial^4\psi}{\partial x^2\,\partial y^2} + \frac{\partial^4\psi}{\partial y^4} \equiv \nabla^4\psi(x, y) = Re\left[\frac{\partial\psi}{\partial y}\left(\frac{\partial^3\psi}{\partial x^3} + \frac{\partial^3\psi}{\partial y^2\partial x}\right) - \frac{\partial\psi}{\partial x}\left(\frac{\partial^3\psi}{\partial x^2\partial y} + \frac{\partial^3\psi}{\partial y^3}\right)\right]. \tag{2}$$

Equation (2) together with boundary conditions (1c–j) form a non-linear biharmonic-type problem for which an exact solution is not attainable through the classical analytical solution methodologies. Here, on the basis of the ideas behind the generalized integral transform technique[20–26] and recent developments for this class of equations,[28] a hybrid numerical–analytical solution is proposed. The appropriate auxiliary problem is chosen as

$$d^4 X_i(x)/dx^4 = \mu_i^4 X_i(x), \quad 0<x<1, \tag{3a}$$

with boundary conditions

$$X_i(0)=0, \qquad dX_i(0)/dx=0, \tag{3b, c}$$

$$X_i(1)=0, \qquad dX_i(1)/dx=0, \tag{3d, e}$$

which is readily solved to yield the related eigenfunctions[28]

$$X_i(x) = \frac{\cosh\mu_i x - \cos\mu_i x}{\cosh\mu_i - \cos\mu_i} - \frac{\sinh\mu_i x - \sin\mu_i x}{\sinh\mu_i - \sin\mu_i} \tag{4a}$$

and the following transcedental equation for evaluation of the eigenvalues:

$$\cos\mu_i \cosh\mu_i = 1. \tag{4b}$$

Problem (3) above enjoys the orthogonality property

$$\int_0^1 X_i(x)\,X_j(x)\,dx = \begin{cases} 0, & i\neq j, \\ N_i, & i=j, \end{cases} \tag{5}$$

which allows definition of the integral transform pair

$$\bar{\psi}_i(y) = \int_0^1 \tilde{X}_i(x)\,\psi(x, y)\,dx \qquad \text{(transform)}, \tag{6a}$$

$$\psi(x, y) = \sum_{i=1}^{\infty} \tilde{X}_i(x)\,\bar{\psi}_i(y) \qquad \text{(inversion)}, \tag{6b}$$

where

$$\tilde{X}_i = X_i / N_i^{1/2} \tag{6c}$$

is the normalized eigenfunction and the norm is computed from

$$N_i = \int_0^1 X_i^2(x)\,\mathrm{d}x. \tag{6d}$$

Following the formalism in the generalized integral transform technique, (2) is now operated on with $\int_0^1 \tilde{X}_i(x)\,\mathrm{d}x$ to provide

$$\frac{\mathrm{d}^4\bar{\psi}_i}{\partial y^4} + 2\sum_{j=1}^{\infty} D_{ij}\frac{\mathrm{d}^2\bar{\psi}_j}{\mathrm{d}y^2} + \mu_i^4\bar{\psi}_i = Re\left(\int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3}\,\mathrm{d}x + \int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x\,\partial y^2}\,\mathrm{d}x - \int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y\,\partial x^2}\,\mathrm{d}x - \int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}\,\mathrm{d}x\right), \tag{7a}$$

where the untransformed term in the ∇^4-operator results in a coupling infinite summation with coefficients

$$D_{ij} = \int_0^1 \tilde{X}_i \tilde{X}_j''\,\mathrm{d}x, \tag{7b}$$

prime denoting differentiation with respect to x.

The four untransformed terms on the right-hand side of the above equation are evaluated by substituting the inversion formula (6b) for ψ and removing the summations from inside the integrals to yield

$$\int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial x^3}\,\mathrm{d}x = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} A_{ijk}\bar{\psi}_k \frac{\mathrm{d}\bar{\psi}_j}{\mathrm{d}y}, \tag{8a}$$

$$\int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial y}\frac{\partial^3\psi}{\partial y^2\,\partial x}\,\mathrm{d}x = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} B_{ijk}\frac{\mathrm{d}\bar{\psi}_j}{\mathrm{d}y}\frac{\mathrm{d}^2\bar{\psi}_k}{\mathrm{d}y^2}, \tag{8b}$$

$$\int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial x^2\,\partial y}\,\mathrm{d}x = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} C_{ijk}\bar{\psi}_j \frac{\mathrm{d}\bar{\psi}_k}{\mathrm{d}y}, \tag{8c}$$

$$\int_0^1 \tilde{X}_i \frac{\partial\psi}{\partial x}\frac{\partial^3\psi}{\partial y^3}\,\mathrm{d}x = \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} B_{ijk}\bar{\psi}_k \frac{\mathrm{d}^3\bar{\psi}_j}{\mathrm{d}y^3}, \tag{8d}$$

where the coefficient matrices are given by

$$A_{ijk} = \int_0^1 \tilde{X}_i \tilde{X}_j \tilde{X}_k'''\,\mathrm{d}x, \tag{9a}$$

$$B_{ijk} = \int_0^1 \tilde{X}_i \tilde{X}_j \tilde{X}_k'\,\mathrm{d}x, \tag{9b}$$

$$C_{ijk} = \int_0^1 \tilde{X}_i \tilde{X}_j' \tilde{X}_k''\,\mathrm{d}x. \tag{9c}$$

All such integrals in (7b) and (9a–c) are readily obtainable in analytical form for best computational performance or accurately and automatically computed through adaptive quadrature.[27]

The transformed biharmonic equation now assumes the form

$$\frac{d^4\bar{\psi}_i}{dy^4} = -\mu_i^4\bar{\psi}_i(y) + \sum_{j=1}^{\infty}\left[-2D_{ij}\frac{d^2\bar{\psi}_j}{dy^2} + Re\sum_{k=1}^{\infty}\left(A_{ijk}\frac{d\bar{\psi}_j}{dy}\bar{\psi}_k + B_{ijk}\frac{d\bar{\psi}_j}{dy}\frac{d^2\bar{\psi}_k}{dy^2} - C_{ijk}\bar{\psi}_j\frac{d\bar{\psi}_k}{dy} - B_{ijk}\frac{d^3\bar{\psi}_k}{dy^3}\bar{\psi}_j\right)\right] \quad (10a)$$

for $i = 1, 2, \ldots, \infty$. Similarly, the boundary conditions at $y = 0$ and 1 can be integral transformed through the same operator to furnish

$$\bar{\psi}_i(0) = 0, \qquad \frac{d\bar{\psi}_i(0)}{dy} = 0, \quad (10b, c)$$

$$\bar{\psi}_i(1) = 0, \qquad \frac{d\bar{\psi}_i(1)}{dy} = \bar{f}_i, \quad (10d, e)$$

where

$$\bar{f}_i = -\int_0^1 \tilde{X}_i \, dx \quad (10f)$$

or

$$\bar{f}_i = \begin{cases} -(4/\mu_i) \quad \tan(\mu_i/2) & \text{for} \quad i \text{ odd}, \\ 0 & \text{for} \quad i \text{ even}. \end{cases} \quad (10g)$$

System (10) forms an infinite set of non-linear fourth-order ordinary differential equations with boundary conditions at two points. For computational purposes this infinite system is truncated to a sufficiently large finite order N to achieve the required error criterion, which corresponds to truncating all the infinite summations involved at the Nth term. The formal aspects behind the convergence of the truncated system solution to the original infinite system solution as $N \to \infty$ have been considered elsewhere.[20,22] From a more practical point of view it suffices to increase the value of N and observe the convergence behaviour. The boundary value problem can be handled numerically by making use of reliable solvers available in well known scientific subroutine libraries,[24] which provide automatic error control schemes, to achieve a user-prescribed accuracy. It suffices to rewrite the truncated system of $4N$ equations in normal form as

$$\mathbf{w}'(y) = \mathbf{p}(\mathbf{w}, y), \quad (11a)$$

where

$$\mathbf{w} = \left[\bar{\psi}_1, \ldots, \bar{\psi}_N, \frac{d\bar{\psi}_1}{dy}, \ldots, \frac{d\bar{\psi}_N}{dy}, \frac{d^2\bar{\psi}_1}{dy^2}, \ldots, \frac{d^2\bar{\psi}_N}{dy^2}, \frac{d^3\bar{\psi}_1}{dy^3}, \ldots, \frac{d^3\bar{\psi}_N}{dy^3}\right]^T, \quad (11b)$$

$$\mathbf{p}(\mathbf{w}, y) = w_{N+l}, \; 1 \le l \le 3N, \quad (11c)$$

$$\mathbf{p}(\mathbf{w}, y) = -\mu_{l-3N}^4 w_{l-3N} + \sum_{j=1}^{N}\left[-2D_{l-3N,j}w_{2N+j} + Re\sum_{k=1}^{N}\left(A_{l-3N,j,k}w_{N+j}w_k + B_{l-3N,j,k}w_{N+j}w_{2N+k} - C_{l-3N,j,k}w_j w_{N+k} - B_{l-3N,j,k}w_k w_{3N+j}\right)\right]. \quad 3N+1 \le l \le 4N, \quad (11d)$$

with boundary conditions

$$w_l(0)=0, \qquad w_{N+l}(0)=0, \qquad l=1,2,\ldots,N, \tag{11e, f}$$

$$w_l(1)=0, \qquad w_{N+l}(1)=\bar{f}_l, \qquad l=1,2,\ldots,N. \tag{11g, h}$$

Once the transformed streamfunctions have been obtained from the numerical solution of system (11), the inversion formula (6b) is recalled to provide an explicit analytical solution for the original potential for any desired position of the integral-transformed co-ordinate. Velocity fields are also readily computed through the appropriate derivatives, in analytical form, of the streamfunction relation.

RESULTS AND DISCUSSION

The present procedure was implemented on a VAX8810 computer and system (11) was handled through subroutine DBVPFD of the IMSL library.[27] A relative error target of 10^{-4} was employed throughout the computations and a few comparative runs were made with tolerances of 10^{-3} and 10^{-5}. The required integrals were evaluated both analytically and numerically through subroutine DQDAGS,[27] for comparison purposes, and no sensible differences were encountered in either precision or computational cost.

First, the convergence behaviour of the present solutions with prescribed accuracy was investigated by varying the number of terms retained in the eigenfunction expansions, N. Different values of Reynolds number which appear more frequently in the literature were considered for critical comparisons, namely $Re=0$, 100 and 400, with truncation orders $N\leq 21$. Fully converged results are therefore expected to be correct to within ± 1 in the fourth significant digit for a relative error target of 10^{-4}.

Figures 2(a)–2(c) show the streamfunction profiles at the centreline of the cavity, $x=0{\cdot}5$, for $Re=0$, 100 and 400 respectively and different truncation orders, $N=6$, 9, 12, 15, 18 and 21. All

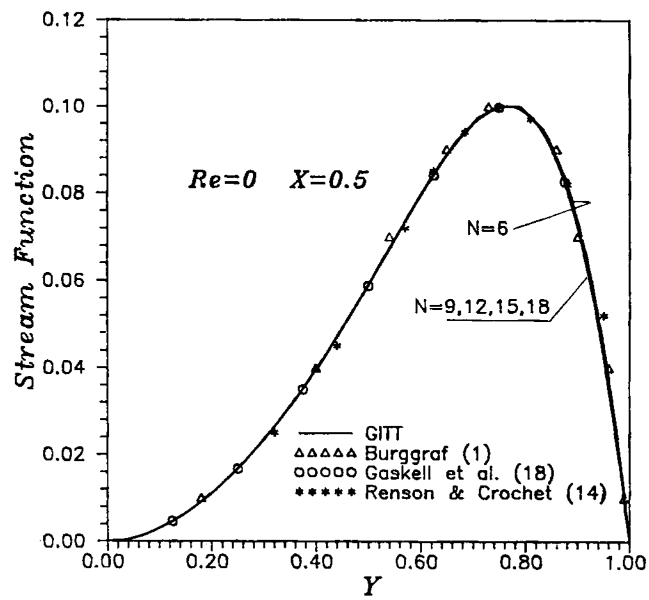

Figure 2(a). Convergence behaviour of streamfunction distribution at cavity centreline, $x=0{\cdot}5$, with $Re=0$

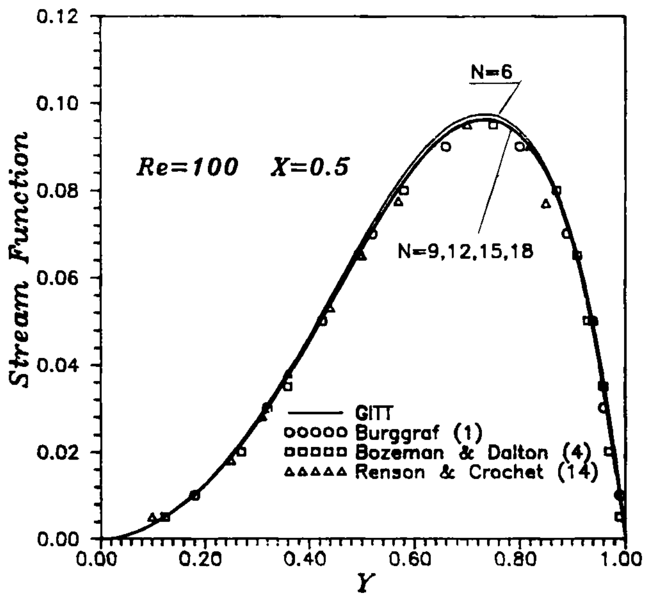

Figure 2(b). Same as Figure 2(a) but with $Re=100$

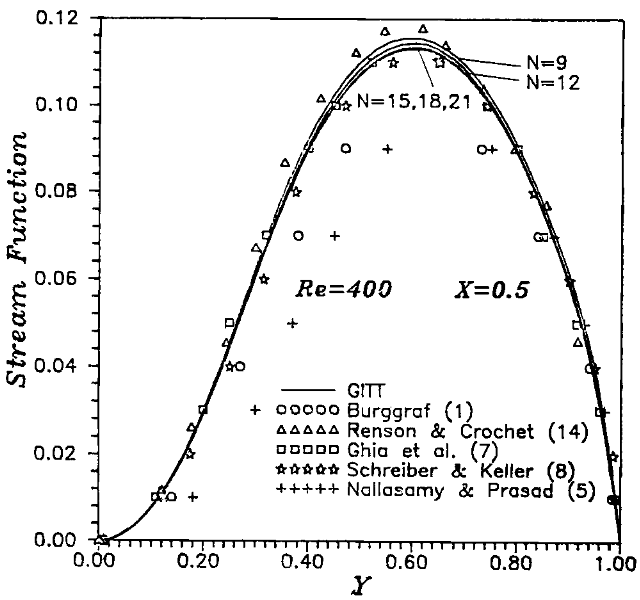

Figure 2(c). Same as Figure 2(a) but with $Re=400$

three graphs indicate that the curves are practically coincident for $N=12$, 15, 18 and 21, with decreasing convergence rates for increasing Reynolds number, as expected, since the more non-homogeneous the biharmonic equation becomes. Also shown are some results from previously reported purely numerical approaches for comparison purposes. Clearly, the early finite differences results of Burggraf[1] become increasingly inaccurate for higher Re, and even more recent

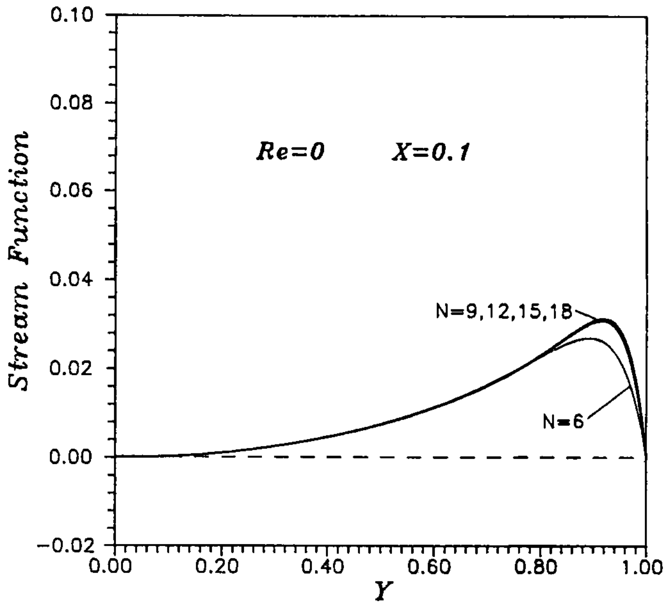

Figure 3(a). Convergence behaviour of streamfunction distribution at $x=0{\cdot}1$ with $Re=0$

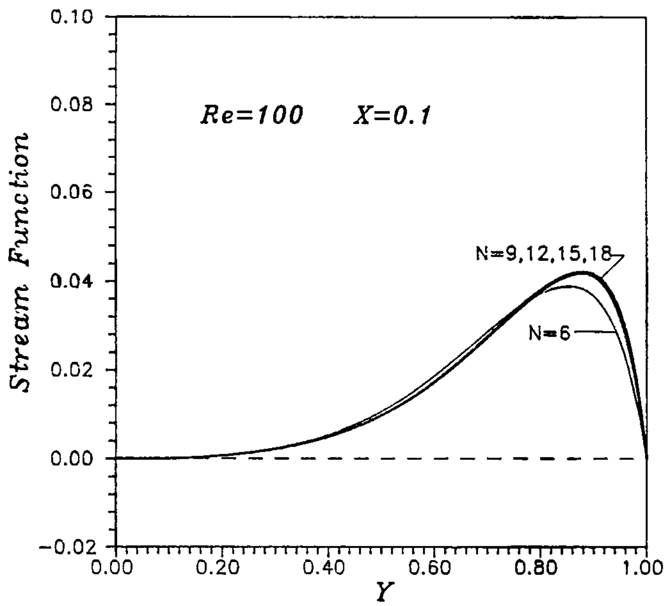

Figure 3(b). Same as Figure 3(a) but with $Re=100$

contributions on finite elements with quadratic elements[14] are still reasonably inaccurate for moderate Reynolds numbers. The best agreement is achieved by efficient finite difference schemes represented by the recent works of Ghia *et al.*[7] and Schreiber and Keller.[8] The results for $Re=400$ clearly indicate that difficulties were encountered in the solution reported by Nallasamy and Krishna Prasad.[5]

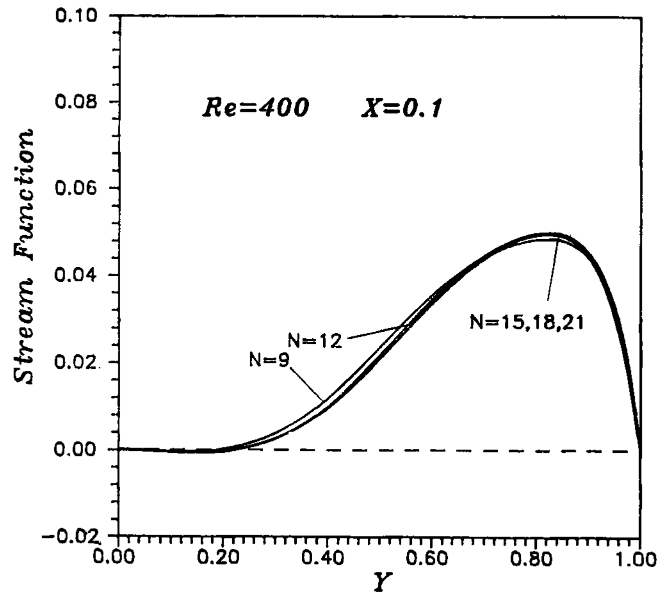

Figure 3(c). Same as Figure 3(a) but with $Re=400$

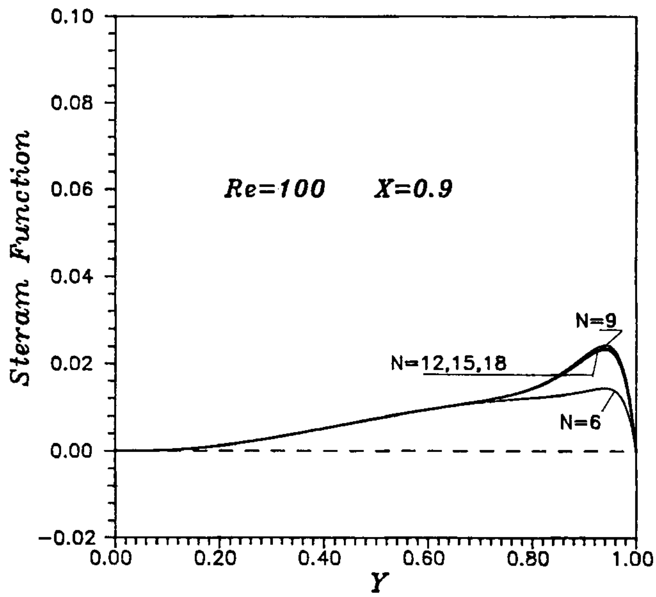

Figure 4(a). Convergence behaviour of streamfunction distribution at $x=0{\cdot}9$ with $Re=100$

An analysis is now performed on the behaviour of the proposed eigenfunction expansion solution for regions in the vicinity of the cavity walls. For instance, Figures 3(a)–3(c) show the streamfunction distributions along the vertical line $x=0{\cdot}1$, again for $Re=0$, 100 and 400 and different truncation orders. Once again the convergence behaviour is a direct function of the relative magnitude of the convection terms, dictated by the value of Re. In all three cases the

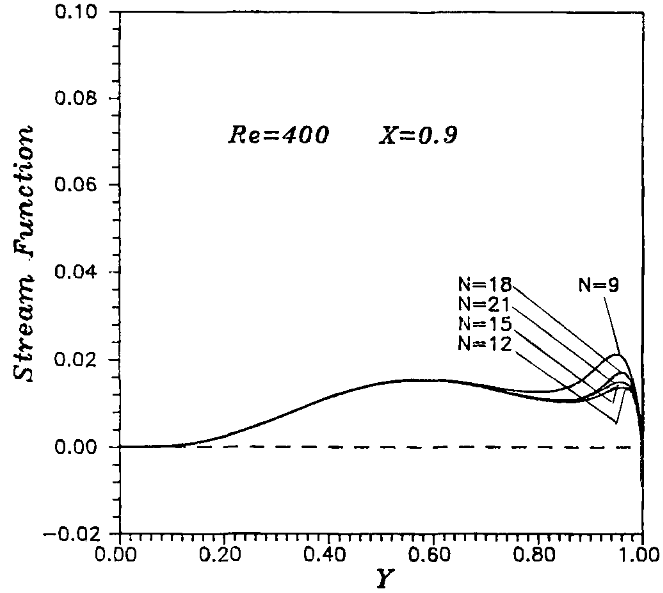

Figure 4(b). Same as Figure 4(a) but with $Re = 400$

results are practically converged for $N > 12$, as in the previous situation of $x = 0{\cdot}5$. Figures 4(a) and 4(b) present the streamfunction profiles in the vicinity of the opposite side wall, along the line $x = 0{\cdot}9$, for $Re = 100$ and 400. The case of $Re = 0$ is not repeated here since it is identical to Figure 3(a) for $x = 0{\cdot}1$ because of symmetry. For $Re = 100$ convergence in the full y-range is achieved for $N > 12$, while for $Re = 400$ truncation orders over 15 were required because of the behaviour close to the corner with the sliding wall.

The same direct expansions evaluated here can be employed for higher Reynolds numbers provided that sufficiently large truncation orders are considered, the price being paid in terms of increased storage and CPU time. Alternatively, one can extract information from the 'source function' represented by the convection terms, making the non-homogeneous part of the biharmonic-type equation less significant. This is accomplished by separating from the original potential a particular simpler solution that includes the source terms, as proposed in References 20 and 24 for Burgers-type equations.

APPENDIX: NOMENCLATURE

A_{ijk}	integral defined by (9a)
B_{ijk}	integral defined by (9b)
C_{ijk}	integral defined by (9c)
D_{ij}	integral defined by (7b)
d	size of square cavity
$\bar{f}_i$	transformed boundary condition (10f)
N	number of terms in truncated eigenfunction expansions
N_i	normalization integral
Re	Reynolds number ($= Ud/\nu$)
U	velocity of top end wall
X_i	eigenfunctions
x, y	dimensionless space co-ordinates

Greek letters

ω	vorticity
ψ	streamfunction
μ_i	eigenvalues
$\bar{\psi}_i$	transformed streamfunction
ν	kinematic viscosity

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